Dirac and Maxwell's Equations Derived from the Density Functions

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Abstract

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We present linear transformations between the components of the electric and magnetic fields, $E = F_1(H)$ and $H = F_2(E)$, which leave the expression for the density of electromagnetic field energy unchanged. If the coefficients occurring in these transformations are interpreted as the components of the momentum vector of the photon and use is made of the correspondence principle $W \to \partial t$ and $\boldsymbol{p} \to \partial \boldsymbol{x}$, then these transformations turn out to be Maxwell's equations. In a similar fashion, when starting from the probability density associated with the Dirac wave function, then one can also arrive, via appropriate transformations between the components of the 4-spinor, at the Dirac equation.

Maxwell's equation were first written in the Dirac form by Majorana by using a wave function the terms of which are the components of the electric and magnetic fields [1], [2], [3]. Since that time many workers have used this excellent idea and reformulated Maxwell's equations in the spinor [4] or quaternionic forms [5], [6]. This points out that a Maxwell-Dirac isomorphism exists which has been described in the works of Sallhofer [7]. It was also found that if we put $m_0 = 0$ in the Dirac equation, $(\gamma_i p^i)\psi = 0$, and associate the following electromagnetic field variables with the components of the 4-spinor ψ [8]:

$$\psi_1 = iE_2, \quad \psi_2 = i(E_1 - iE), \quad \psi_3 = H_3 \quad \text{and}$$
$$\psi_4 = H_1 + iH_2, \tag{1}$$

then we arrive at Maxwell's equations

$$\nabla \times \boldsymbol{H} = \boldsymbol{E} c^{-1}, \quad \nabla \cdot \boldsymbol{H} = 0, \tag{1a}$$
$$\nabla \times \boldsymbol{E} = -\boldsymbol{H} c^{-1}, \nabla \cdot \boldsymbol{E} = 0. \tag{1b}$$

Equally well we can arrive at (1a) and (1b) if we take the conjugate Dirac equation and put

$$\tilde{\psi}_1 = -iE_2, \quad \tilde{\psi}_2 = -i(E_1 - iE_2), \quad \tilde{\psi}_3 = H_3 \quad \text{and}$$
$$\tilde{\psi}_4 = H_1 - iH_2.$$

The bilinear form, $(8\pi)^{-1}\psi_i\tilde{\psi}^i$, represents the density of the electromagnetic field energy

$$e(x, t) = \frac{1}{8\pi}(E_1^2 + E_2^2 + E_3^2 + H_1^2 + H_2^2 + H_3^2). \tag{2}$$

Next we shall show that it is possible to proceed in the reverse order, namely, to start from the expression for the density of the electromagnetic energy (2) and arrive at Maxwell's equations via those linear transformations between the components of $\boldsymbol{E}$ and $\boldsymbol{H}$ which leave the expression (2) unchanged, the coefficients occurring in these transformations being considered as the components of the momentum 4-vector of photon. In a similar fashion, taking the probability density associated with the Dirac wave function we can also arrive, via the appropriate transformations between the components of the 4-spinor, ψ, at the Dirac equation.

Let us now look for the linear transformations of the form

$$E_i = F_i(H_1, H_2, H_3) \quad \text{and} \quad H_i = F_i(E_1, E_2, E_3) \tag{3}$$

which leave expression (2) unchanged. The simplest but trivial transformations which meet this requirement have the form $\boldsymbol{E} = \boldsymbol{H}$ and $\boldsymbol{H} = \boldsymbol{E}$. By means of the direct substitution in (2) one can prove that the linear and non-trivial transformations which leave (2) unchanged are

$$\begin{aligned} aE_1 &= b_3H_2 - B_2H_3, \\ aE_2 &= -b_3H_1 + b_1H_3, \\ aE_3 &= b_2H_1 - b_1H_2, \end{aligned} \tag{4a}$$

$$\begin{aligned} aH_1 &= -b_3E_2 + b_2E_3, \\ aH_2 &= b_3E_1 - B_1E_3, \\ aH_3 &= -b_2E_1 + b_1E_2. \end{aligned} \tag{4c}$$

To these equations we must add the transversibility conditions [1], [2]

$$0 = -b_1H_1 - b_2H_2 - b_3H_3 \tag{4b}$$ and
$$0 = b_1E_1 + b_2E_2 + b_3E_3. \tag{4d}$$

In (4a), (4b), (4c) and (4d) the equation $a^2 = b_1^2 + b_2^2 + b_3^2$ is to be satisfied. If we multiply (4c) and (4d) by $-i$ and add to them (4a) and (4b), we can write the transformations (4) in the compact form

$$\begin{aligned} a\boldsymbol{\psi} + i\boldsymbol{b} \times \boldsymbol{\psi} &= 0, \\ i\boldsymbol{b} \cdot \boldsymbol{\psi} &= 0, \end{aligned} \tag{5}$$

where $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3)$, $\psi_1 = E_1 - iH_1$, $\psi_2 = E_2 - iH_2$, $\psi_3 = E_3 - iH_3$ and $\boldsymbol{b} \equiv (b_1, b_2, b_3)$. If we interpret the coefficients a, b_1, b_2, b_3 as the components of the momentum 4-vector of the photon, i.e. if we put $a = Wc^{-1}$, $b_1 = p_1$, $b_2 = p_2$, $b_3 = p_3$ and use the correspondence principle, $W \to \partial t$ and $\boldsymbol{p} \to \partial \boldsymbol{x}$, equations (5) become Maxwell's equations in the form written first by Majorana [1].

Now let us proceed in a similar way in the case of the Dirac equation. As is well known, the probability density of the Dirac wave function has the form

$$\varrho(x, t) = \chi_1^2 + \chi_2^2 + \chi_3^2 + \chi_4^2 + \Phi_1^2 + \Phi_2^2 + \Phi_3^2 + \Phi_4^2, \tag{6}$$

where χ_1, χ_2, χ_3, χ_4 and Φ_1, Φ_2, Φ_3, Φ_4 are the functions occurring in the components of the 4-spinor ψ. The most general linear transformations between the components χ_1, χ_2, χ_3, χ_4 and Φ_1, Φ_2, Φ_3, Φ_4, which leave the expression (6) unchanged, have the form

$$\begin{aligned} a\chi_1 &= b_0\Phi_1 + b_3\Phi_2 - b_3\Phi_3 + b_1\Phi_4, \\ a\chi_2 &= -b_3\Phi_1 + b_0\Phi_2 + b_1\Phi_3 + b_2\Phi_4, \\ a\chi_3 &= b_2\Phi_1 - b_1\Phi_2 + b_0\Phi_3 + b_3\Phi_4, \\ a\chi_4 &= -b_1\Phi_1 - b_2\Phi_2 - b_3\Phi_3 + b_0\Phi_4 \end{aligned} \tag{7a}$$

and

$$\begin{aligned} a\Phi_1 &= b_0\chi_1 - b_3\chi_2 + b_2\chi_3 - b_1\chi_4, \\ a\Phi_2 &= b_3\chi_1 - b_0\chi_2 - b_1\chi_3 - b_2\chi_4, \\ a\Phi_3 &= -b_2\chi_1 + b_1\chi_2 + b_0\chi_3 - b_3\chi_4, \\ a\Phi_4 &= b_1\chi_1 + b_2\chi_2 + b_3\chi_3 + b_0\chi_4, \end{aligned} \tag{7b}$$

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whereby the condition $a^2 = b_1^2 + b_2^2 + b_3^2 + b_0^2$ is to be satisfied. If we again interpret the coefficients a, b_1, b_2, b_3 as the components of the momentum of the particle with the positive rest mass m_0 (Wc^{-1}, p_1, p_2, p_3) and choose the following combinations of the functions $\chi_1, \chi_2, \chi_3, \chi_4$ and $\Phi_1, \Phi_2, \Phi_3, \Phi_4$, for the components of the 4-spinor ψ: $\psi_1 = i\chi_3 + \chi_4$, $\psi_2 = i(\chi_1 + i\chi_2)$, $\psi_3 = \Phi_3 - i\Phi_4$ and $\psi_4 = \Phi_1 + i\Phi_2$, then we may write the transformations (7) in the compact form

$$\left(-\frac{W}{c} + \sum_{i=1}^{3} \gamma_i p^i + \gamma_0 m_0 c\right)\psi = 0 \ , \tag{8}$$

where

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are matrices which satisfy the relations

$$\gamma_i \gamma_k + \gamma_k \gamma_i = \delta_{ik} \ ; \quad i, k = 0, 1, 2, 3 \ .$$

When replacing the components of the momentum vector by the corresponding operator (8) becomes the Dirac equation.

Thus it is possible to obtain Maxwell's equations and Dirac's equation from such linear transformations between the field components and the components of the 4-spinor which leave the expression for the density of the electromagnetic field and the probability density of the Dirac field unchanged, respectively when we interpret the coefficients in these transformations as the components of the momentum 4-vector and of a massless or massive particle, and apply the correspondence principle. It is interesting to investigate what kinds of field equations arise if we combine different field variables with the momentum 4-vector for massless and massive particles. Here four different cases can occur: (i) The connection of the electromagnetic field with the 4-momentum for the massless particle leads to standard Maxwell's equations; (ii) The connection of the classical electromagnetic field with 4-momentum for the massive particle leads to the "massive" variant of Maxwell's equations which we obtain when inserting (1) into (8):

$$\begin{aligned} \boldsymbol{H}c^{-1} &= \nabla \times \boldsymbol{E} + k_0 \boldsymbol{E} \ ; \quad \nabla \cdot \boldsymbol{E} = 0 \ , \quad k_0 = m_0 c/\hbar \ , \\ \boldsymbol{E}c^{-1} &= \nabla \times \boldsymbol{H} + k_0 \boldsymbol{H} \ ; \quad \nabla \cdot \boldsymbol{H} = 0 \ . \end{aligned} \tag{9}$$

After eliminating $\boldsymbol{E}$ or $\boldsymbol{H}$ we have

$$(\Box + k_0^2)\boldsymbol{E} = 0 \quad \text{and} \quad (\Box + k_0^2)\boldsymbol{H} = 0 \ ,$$

respectively; (iii) The Dirac field and the momentum for the massive particle lead to the standard form of the Dirac equation; (iv) The Dirac field and 4-momentum for the massless particle lead to the Weyl equation.

A special case occurs if we associate with the components of the 4-spinor ψ the field variables

$$\begin{aligned} \psi_1 &= iE_3 + q \ , \quad \psi_2 = i(E_1 + iE_2) \ , \\ \psi_3 &= H_3 + ip \ , \quad \psi_4 = H_1 + iH_2 , \end{aligned} \tag{10}$$

where q and p are the new scalar field variables introduced by Ohmura [9]. Inserting (10) into (8) we get the extended Maxwell equations in the form

$$\begin{aligned} \boldsymbol{E}c^{-1} - \nabla \times \boldsymbol{H} + \nabla p = 0 \ , \quad pc^{-1} + \nabla \cdot \boldsymbol{E} = 0 \ , \\ \boldsymbol{H}c^{-1} + \nabla \times \boldsymbol{E} + \nabla q = 0 \ , \quad qc^{-1} + \nabla \cdot \boldsymbol{H} = 0 \ . \end{aligned}$$

These field equations have been used as the linear field equations in the theory of gravitation [10], [11], [12].

From what has been said so far it follow that all important linear equations in physics can be derived from the transformations between the field components, which leave certain of their quadratic forms unchanged, when we appropriately interpret the coefficients occurring in them.

This can serve as a *guiding heuristic* principle for *finding* new mathematical equations for the different physical fields.

[1] E. Majorana, Scientific manuscripts, unpublished, deposited at the "Domus Galilacana", Pisa: Quaderno 2, p. 101; 3, p. 11, 160; 15, p. 16; 17, p. 83, 159 (1928 – 1932).
[2] R. Mignani, E. Recami, and M. Baldo, Lett. Nuovo Cim. **11**, 568 (1974).
[3] E. Giannetto, Nuovo Cim. **44**, 140 (1985).
[4] A. A. Campolattaro, Int. J. Theor. Phys. **19**, 99, 127 (1980).
[5] J. D. Edmonds Jr, Lett. Nuovo Cim. **13**, 185 (1975).
[6] V. Majerník and M. Nagy, Lett. Nuovo Cim. **16**, 256 (1976).
[7] H. Sallhofer, Z. Naturforsch. **41a**, 1087 (1986) and all references herein.
[8] C. Schaefer, Einführung in die theoretische Physik, Band III. W. de Gruyter, Berlin 1951, p. 456.
[9] T. Ohmura, Prog. Theor. Phys., **16**, 684 (1965).
[10] A. Da Silveira, Lett. Nuovo Cim. **35**, 341 (1982).
[11] A, Singh, Lett. Nuovo Cim. **33**, 457 (1982).
[12] V. Majernik, Astrophys. Space Sci. **84**, 191, and **82**, 473 (1982).